AD-A237 285

REPORT DOCUMENTATION PAGE

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	May 1991	professional paper

4. TITLE AND SUBTITLE

DEDUCTION AND INFERENCE USING CONDITIONAL LOGIC AND PROBABILITY

5. FUNDING NUMBERS

In-house

6. AUTHOR(S)

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7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Naval Ocean Systems Center
San Diego, CA 92152-5000

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Naval Ocean Systems Center
San Diego, CA 92152-5000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT	12b. DISTRIBUTION CODE
Approved for public release; distribution is unlimited.	

13. ABSTRACT (Maximum 200 words)

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Published in *Conditional Logic in Expert Systems* by I. R. Goodman, H. T. Nguyen, M. M. Gupta, and G. S. Rogers, eds. 1991. North Holland Press.

14. SUBJECT TERMS			15. NUMBER OF PAGES
conditional propositions	logic	deduction	
conditional events	reasoning with uncertainty		16. PRICE CODE
inference	3-valued logic	conditional probability	

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAME AS PAPER

21a NAME OF RESPONSIBLE INDIVIDUAL	21b TELEPHONE *(Include Area Code)*	21c OFFICE SYMBOL
P. G. Calabrese	(619) 553-4042	Code 421

In *Conditional Logic in Expert Systems* by I. R. Goodman, H. T. Nguyen, M. M. Gupta and G. S. Rogers, eds. (accepted for publication in 1991 by North-Holland Press)

DEDUCTION AND INFERENCE USING CONDITIONAL LOGIC AND PROBABILITY

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Abstract: *In contrast to the autnor's 1987 paper, which presented an algebraic synthesis of conditional logic and conditional probability starting with an initial Boolean algebra of propositions, this paper starts with an initial probability space of events and generates the associated propositions as measurable indicator functions (à la the approach of B. De Finetti). Conditional propositions are generated as measurable indicator functions restricted to subsets of positive probability measure. The operations of "and", "or", "not" and "given" are defined for arbitrary conditional propositions. The representation of the resulting conditional event algebra as a 3-valued logic (always possible according to a new theorem due to I. R. Goodman) is given in terms of 3-valued truth tables. Formulas for the conditional probability of complex conditional propositions such as (q|p) ∨ (s|r) are proved. A second major theme of the paper concerns deduction in the realm of conditional propositions. It turns out that their are varieties of logical deduction for conditional propositions depending on the particular entailment relation (≤) chosen. These relations are explored including their lattice properties and properties of non-monotonicity. Computational aspects for Artificial Intelligence are also discussed.*

Keywords: Conditional propositions, conditional events, logic, reasoning with uncertainty, 3-valued logic, conditional probability, deduction, inference

1. Introduction

Deep within the foundations of logic and probability, the architects and builders have left a missing stone. Roughly, this foundation stone is to logical propositions what fractions are to integers. Now, with the advent of the computer age, attempts to incorporate more of human intelligence into machines (so-called artificial intelligence) have exposed this lack of foundation and led computer scientists to resort to sub-optimal methods to compute actions from information via some "reasonable" data fusion algorithm. Hence there is no

standard theory for combining information in the context of uncertainty. Among the partially overlapping techniques there are:

a. The *fuzzy sets* and other fuzzy language modifiers and methods of L. Zadeh [5],

b. The *belief functions* of Dempster-Shafer [7], and

c. The *probability logic* approach of P. Calabrese [2] and [3].

The author will leave it to the many enthusiasts of fuzziness to crystallize the imprecise and generally wasteful information combining techniques commonly employed by the common man as he commonly goes bumbling through life. This is not to say that we do not need approximate methods by which to combine information in the face of the default of logic and probability to provide more precise methods. Even though fuzzy methods tend to distort information at least these methods come up with solutions, and often an exact solution is not necessary. Nevertheless, science should continually seek to purge all unnecessary natural language ambiguities from its formal mathematical descriptions, not meekly incorporate them! A new theory should also, if possible, merge with the older theory where the older is tested and applicable. *That* the fuzzy approach does not do. Before one adopts a distorting technique, no matter how computationally tractable it may be, one should first extend the classical theories of logic and probability as far as possible, and secondly, merge with them on the boundary of their domain of application. However, except for a few authors (for example, J. Pearl [6] and his important work in conditional independence) this has not been attempted by the new generation of uncertainty workers in so-called artificial intelligence. Instead, many researchers have publicly discounted the practicality of probability theory as a method for reasoning with uncertainty - a notion that has prompted P. Cheeseman [4] to make a "defense of probability theory".

Another technique for reasoning with uncertainty is the belief function approach of Dempster-Shafer [7] which, while striving to be consistent with probability theory, addresses the problem of determining the support for propositions arising from even mutually inconsistent evidence.

The third approach to dealing with uncertainty is actually the oldest. G. Boole himself, the father of the algebra of logic, was developing an algebra of logic *and* probability (see T. Hailperin's cogent account [8]) but he died before completing the work. His unfinished algebraic development was then abbreviated by his successors, who attached his name to the resulting algebra.

In 1932, 1934 and later in 1956, S. Mazurkiewicz [9], [10] and [11] used A. Tarski's [12], [13] new theory of algebraic logic to approach the problem of conditioning in an

algebraic setting, but he did not get very far before his death. At the same time N. Kolmogorov [28] was laying down his successful axiomatization of probability theory and he realized that he could not follow logic in equating "if p then q" to "q or not p".

Already, in 1913, B. Russell and A. N. Whitehead [1] had made truth tables and so-called material implication the standard form of implication in logic, and this worked fairly well for 2-valued logic, but Kolmogorov found it to be inappropriate for probability theory. It has also been known at least since 1975, [2] and [3], that the probability $P(q \vee p')$ of the material conditional is, in general, greater than the conditional probability $P(q|p)$ of q given p, unless either $P(p) = 1$ or $P(q|p) = 1$. Furthermore, if $p = 0$ then $q \vee p'$ is certain (=1) but $P(q|p)$ is undefined. This telltale inadequacy of material implication for representing "if - then - " has been noticed by generations of introductory logic students who have questioned why "if p then q" should be true or "valid" in case p is false. This question by pre-indoctrinated logic students has all too often been squelched by their instructors, who blithely appealed to the assignment of exactly *two* truth values to show that "if p then q" must be equivalent to "q or not p". Consequently if p is false then "not p" is true, and so too is $(q \vee p')$, whatever the truth value of q! Thus (the argument goes) "if p then q" is true (valid) when p is false.

Nevertheless, a good scientist does not include cases in his sample for which the premise of his hypothesis is false; he does not count such cases as positive evidence of his hypothesis irrespective of the truth of his conclusion. Nor does a scientist report the probability that either the conclusion of his hypothesis is true or its premise false; rather, he reports the conditional probability of the conclusion of his hypothesis given that its premise is true; and so too must those who would consistently quantify the truth content of partially true statements.

Besides this divergence between the treatments of "if - then -" in the domains of logic versus probability, there also tends to be an inadequate distinction made in logic between propositions that are partially true and propositions that are wholly true. Generally, in a Boolean algebra a proposition need not be either true in all models (interpretations, worlds) or false in all models; a proposition can be true in some and false in others, thus allowing it to have a non-trivial probability. Nevertheless, the lack of a commonly accepted algebraic context for both logic and probability has made the very meaning of the "probability of a proposition" controversial. This is true in spite of the fact that G. Boole [29], R. Carnap and R. C. Jeffrey [30] & [31], H. Gaifman [32], D. Scott and P. Kraus [33], E. W. Adams [19], and T. Hailperin [8] have all defined the probability of a

proposition as the probability of its extension set of models, i.e., the probability of the set of models (interpretations, worlds) in which the proposition is true.

Others who have contributed to the expansion of probability logic that should be mentioned include B. De Finetti [14], who first treated propositions as indicator functions from a sample space to {0,1}; P. Rosenbloom [15], whose treatment of algebraic logic was very influential to the author; G. Schay [16], who was probably the first person to define a system of conditional propositions that included operations for combining propositions with different premises; N. Rescher [17], whose monumental 1969 book *Many Valued Logics* (still the standard in the field) included the 3 valued logic of B. Sobocinski [18], which turns out to be equivalent to the author's system less conditional conditionals; E. W. Adams [19], whose operations are equivalent to those of Sobocinski; D. Dubois and H. Prade [20], who have carefully reviewed the recent literature and contrasted the author's conditional logic from that of I. R. Goodman & H. T. Nguyen; and finally I. R. Goodman & H. T. Nguyen, who upon reading an early (1986) manuscript of the author's 1987 paper, immediately realized the crucial importance of conditional events, conducted a comprehensive historical review concerning the problem of conditioning [21], and later contributed to the algebraic foundations of conditionals, initiated new directions for research and discovered significant new results [22]. (I would like to thank these colleagues for discovering the work of G. Schay and B. Sobocinski, and for pointing out similarities between the author's system and those of Schay, Sobocinski and Adams.)

The next section begins with a probability space and defines propositions (à la B. De Finetti [14]) as indicator functions defined on the elementary set of occurrences of a probability space. The meaning of a proposition being partially true or wholly true is defined in the context of the algebraic logic of propositions (see, for instance, Chang and Keisler [26].) The probability of each proposition is then defined in terms of a probability measure on the extensionally associated models (interpretations) that satisfy those propositions.

Conditional propositions (q|p), "q given p", are next defined as domain-restricted P-measurable indicator functions which can be combined by "and", "or", "not" and "given" resulting in another such conditional proposition. The resulting system of conditionals can be represented as a 3-valued logic, as predicted by a recent theorem of I. R. Goodman [22, and this book]. The third value does not represent uncertainty but rather *inapplicability* - falseness of the premise of the conditional proposition. (Uncertainty is automatically represented by non-atomic propositions, that thereby leave various possible facts unspecified.) A new formula is given for the probability of the disjunction, $(q|p) \vee (s|r)$, of two conditional expressions, thereby generalizing the well-known formula $P(q \vee p) =$

$P(q) + P(p) - P(q \wedge p)$. A non-trivial formula for the conjunction of two conditionals is also proved.

In the subsequent section on deduction, two types of deduction in a Boolean algebra are distinguished. One of these types splits into four non-equivalent types of deduction in the realm of conditionals resulting in at least five different kinds of deduction. These types of deduction are characterized in terms of relationships between the original unconditioned propositions.

2. Formal Development

Propositions, Probability Spaces and Indicator Functions: If $\mathcal{P} = (\Omega, \mathcal{B}, P)$ is a probability space then the characteristic function of each P-measurable subset B, $B \in \mathcal{B}$, defines a unique P-measurable indicator function q: $\Omega \rightarrow \{0,1\}$ from Ω to the 2-element Boolean Algebra $\{0,1\}$ as follows:

$$q(\omega) = \begin{cases} 1, \text{ if } \omega \in B, \\ 0, \text{ if } \omega \in B' \end{cases} \tag{1}$$

q is a "proposition" in the sense that for each $\omega \in \Omega$, either q is true for ω (i.e. $q(\omega) = 1$) or q is false for ω (i.e. $q(\omega) = 0$). Let L denote the set of all propositions of $\mathcal{P}$.

Conversely, each P-measurable indicator function q defines a unique P-measurable subset B, $B \in \mathcal{B}$ by

$$B = q^{-1}(1) = \{\omega \in \Omega: q(\omega) = 1\}. \tag{2}$$

B is the P-measurable subset on which q is true, and P(B) is the probability measure of the partial truth of q, and so $P(q) = P(q^{-1}(1))$.

In this correspondence between measurable subsets (probabilistic events) and measurable indicator functions (propositions) the whole set Ω corresponds to the unity indicator function, to those propositions that are true in all ω --- necessary & provable. The empty set Φ corresponds to the zero indicator function, to those propositions that are false in all ω --- impossible and contradictory.

Definition 1: Two propositions (indicator functions) p and q are equivalent if and only if they are equal as functions. That is, p = q if and only if both p and q take the value 1 (or 0) on the same subset of Ω. Thus p = q if and only if $p^{-1}(1) = q^{-1}(1)$ if and only if $p^{-1}(0) = q^{-1}(0)$.

Axioms of Boolean Algebra: A Boolean algebra, as formulated by T. Hailperin [8], is a set of propositions L (including two constants 0 and 1) that is closed under the three operations "and" (juxtaposition or $\wedge$), "or" ($\vee$) and "not" (') and that satisfies these axioms:

$$\begin{aligned} pq &= qp, & p \vee q &= q \vee p, \\ (pq)r &= p(qr), & (p \vee q) \vee r &= p \vee (q \vee r), \\ (1)(p) &= p, & 0 \vee p &= p, \\ (p)(p') &= 0, & p \vee p' &= 1, \\ p(q \vee r) &= pq \vee pr, & p \vee (qr) &= (p \vee q)(p \vee r), \\ pp &= p, & p \vee p &= p. \end{aligned} \tag{3}$$

Conditional Propositions: In order to incorporate conditions, consider next that each ordered pair, (B|A) of P-measurable subsets B, A in $\mathfrak{B}$ with corresponding pairs (q|p) of indicator functions q, p, defines a unique domain-restrcted P-measurable indicator function (q|p): A $\rightarrow$ {0,1} from A to the 2-element Boolean algebra as follows:

Definition 2:

$$(q|p)(\omega) = \begin{cases} 1, \text{ if } \omega \in (A \cap B), \\ 0, \text{ if } \omega \in (A \cap B'), \\ \text{undefined, if } \omega \in A' \end{cases} \tag{4}$$

In terms of the unconditioned propositions p and q this is

$$(q|p)(\omega) = \begin{cases} q(\omega), \text{ if } p(\omega) = 1, \\ \text{undefined, if } p(\omega) = 0. \end{cases} \tag{5}$$

(q|p) is a "conditional proposition" in the sense that if p is true on ω then (q|p) is either true on ω or false on ω depending on the truth value of q. If p is false on ω, we say that (q|p) does not apply (i.e., is undefined) for ω. (q|p) is q, restricted to $p^{-1}(1)$, the subset on which p is true. The set of all conditional propositions of $\mathcal{P}$ will be denoted L/L.

Conversely, each such ordered pair of P-measurable indicator functions (q|p) defines a unique ordered pair, (B|A), of P-measurable subsets where $A = p^{-1}(1)$ and $B = q^{-1}(1)$. A is the measurable subset on which p is true and B is the measurable subset on which q is true. $B \cap A$ is the measurable subset of A on which q is also true, and for non-zero P(A), $P(B \cap A) / P(A)$ is the conditional probability of q given p, denoted P(q|p).

Boolean Operations: The operations "or" ($\vee$), "and" (juxtaposition or $\wedge$) and "not" ('), defined on the Boolean algebra (or sigma-algebra) $\mathfrak{B}$ of events of $\mathcal{P}$ naturally generate

operations on the indicator functions via disjunction, conjunction and negation in the 2-element Boolean algebra {0,1} as follows:

$$\begin{aligned}(p \vee q)(\omega) &= p(\omega) \vee q(\omega), \\ (pq)(\omega) &= p(\omega)q(\omega), \\ p'(\omega) &= (p(\omega))'.\end{aligned} \tag{6}$$

Here, the operations on the right hand side are in the 2-element Boolean algebra.

Note further that the first two operations can be expressed in terms of the minimum and maximum functions on {0,1}:

$$\begin{aligned}p \vee q &= \max\{p, q\} \\ pq &= \min\{p, q\}\end{aligned} \tag{7}$$

Together with the Boolean axioms and truth assignments the set of propositions L forms a Boolean logic, which will formally be denoted $\mathcal{L}$.

In this framework each probabilistic outcome $\omega \in \Omega$ is a *model* [26, pp. 1-2] of the Boolean logic $\mathcal{L}$ because firstly, the axioms of the Boolean logic $\mathcal{L}$ are true in ω and secondly, ω assigns each *proposition of* $\mathcal{L}$ an unambiguous truth value of true or false.

The above approach to probability logic starts with a probability space $\mathcal{P} = (\Omega, \mathcal{B}, P)$ and generates a Boolean algebra $\mathcal{L}$ of propositions, each proposition of which has a probability. Another possible approach is to assume a probability measure on a given Boolean algebra of propositions and thereby induce a probability measure on the models of that Boolean algebra. Still another way is to assume a probability measure on the models of a given Boolean algebra and induce a measure on the associated propositions. For the latter approach see P. Calabrese [3].

Now it is known that not every Boolean algebra admits a probability measure P. Nor does every σ-algebra $\mathcal{B}$ admit a probability measure P. These pathological cases will not be discussed here. Suffice it to say that if a Boolean algebra is finite or at least atomic then there is no problem establishing a probability measure on it.

Equivalence of Conditional Propositions: Having defined conditional propositions as indicator functions, the equivalence of two conditional propositions is easy to define:

Definition 3: Two conditional propositions $(q|p)$ and $(s|r)$ are equivalent, i.e. $(q|p) = (s|r)$, if and only if they are equal as indicator functions, that is, if and only if they have the same domain and are equal on this common domain.

Theorem 1: Two conditionals $(q|p)$ and $(s|r)$ are equivalent if and only if they have equivalent premises and their conclusions are equivalent in conjunction with that premise. That is, $(q|p) = (s|r)$ if and only if $p = r$ and $qp = sr$.

Proof of Theorem 1: By definition $(q|p) = (s|r)$ if and only if they are functionally equal. The common domain of the indicator functions $(q|p)$ and $(s|r)$ is $p^{-1}(1)$ and $r^{-1}(1)$. So $p = r$. The subset of $p^{-1}(1)$ on which $(q|p)$ equals 1 is $[q^{-1}(1)]\ [p^{-1}(1)] = \{\omega \in \Omega: q(\omega) = 1 \text{ and } p(\omega) = 1\} = \{\omega \in \Omega: (qp)(\omega) = 1\} = (qp)^{-1}(1)$. Similarly, the subset of $r^{-1}(1)$ on which $(s|r)$ is 1 is $(sr)^{-1}(1)$. Since these subsets are equal, $qp = sr$. Conversely, if $p = q$ and $qp = sr$ then $(q|p)$ and $(s|r)$ have the common domain $p^{-1}(1)$. Furthermore, on $p^{-1}(1)$, which is also $r^{-1}(1)$, $(q|p)(\omega) = q(\omega) = q(\omega)\ p(\omega) = (qp)(\omega)$. Similarly, on $r^{-1}(1)$, $(s|r)(\omega) = (sr)(\omega)$. But $qp = sr$. So $(q|p) = (s|r)$.

The equivalence class of conditional propositions containing the conditional proposition $(q|p)$ is $\{(s|r) \in L/L: (s|r) = (q|p)\} = \{(s|r): r = p \text{ and } sr = qp\} = \{(s|p): sp = qp\}$, and may be denoted $<q|p>$, or when there is no ambiguity simply as $(q|p)$. This class or coset of propositions is the set of all domain-restricted indicator functions which agree with q on the subset $p^{-1}(1)$, where p is has the value 1. The coset $<p|p>$ containing $(p|p)$ is just $\{(s|r) \in L/L: (s|r) = (p|p)\} = \{(s|p): sp = p\}$.

Note that the conditionals $\{(q|0): q \in L\}$ form an equivalence class of wholly undefined conditionals --- conditionals that have impossible premises. Note also that for every conditional proposition $(q|p)$, $(q|p) = (qp|p)$.

Definition 4: A conditional proposition $(q|p)$ is said to be in reduced form if $qp = p$.

Note that if $(q|p)$ is in reduced form, then $P(q|p) = P(q)/P(p)$.

It is instructive to note that in general 2 events A, B (or propositions p, q) generate $2^2 = 4$ nonempty atomic events {AB, AB', A'B, A'B'} and $2^4 = 16$ non-equivalent subsets of these atomic events, and $3^4 = 81$ non-equivalent conditional events (conditional propositions) --- all from just two initial binary variables! (For a proof that the number of non-equivalent conditionals is 3^N, where N is the number of atomic events, see [3], p. 225.) Starting with 4 propositions, $2^{16} = 65{,}536$ non-equivalent propositions and $3^{16} = 43{,}046{,}721$ non-equivalent conditional propositions may be generated!

Operating with Undefined Conditionals: The Boolean operations can be extended in various ways to the domain-restricted indicator functions (i.e. to the conditional propositions) depending upon how one regards the effect of undefined conditionals in combination with defined conditionals.

in this paper a conditional that is undefined for a particular ω will (for that ω) have no effect upon any other conditional with which it may be disjoined or conjoined or which it may condition. That is, if $(s|r)(\omega)$ is undefined then $(s|r)(\omega)$ acts like an operational identity with respect to disjunction, conjunction and when acting as a premise. Furthermore, when acting as a conclusion, such a conditional results in an undefined conditional no matter what the premise. This corresponds to the usual way people handle inapplicable conditionals in practice. These assumptions can be expressed succinctly as follows:

Axioms of Conditional Probability Logic: Let c be an arbitrary conditional proposition and let d be a conditional proposition that is undefined on ω, i.e., $d = (s|r)$ where $r(\omega) = 0$:

$$
\begin{aligned}
(d(\omega))' &= d(\omega) \\
c(\omega) \vee d(\omega) &= c(\omega) \\
c(\omega)\, d(\omega) &= c(\omega) \\
c(\omega) \mid d(\omega) &= c(\omega) \\
d(\omega) \mid c(\omega) &= d(\omega)
\end{aligned}
\tag{8}
$$

With this understanding, (also see Dubois & Prade [20]), the extensions of the three operations to conditionals takes the following natural functional form:

Definition 5: For arbitrary conditionals $(q|p)$ and $(s|r)$,

$$
\begin{aligned}
[(q|p) \vee (s|r)](\omega) &= (q|p)(\omega) \vee (s|r)(\omega) \\
[(q|p) \wedge (s|r)](\omega) &= [(q|p)(\omega)] \wedge [(s|r)(\omega)] \\
[(q|p)'](\omega) &= ((q|p)(\omega))'
\end{aligned}
\tag{9}
$$

Theorem 2: In terms of a single conditional of the original propositions q, p, s, and r, these three operations become:

$$
\begin{aligned}
(q|p) \vee (s|r) &= (qp \vee sr) \mid (p \vee r) \\
(q|p)(s|r) &= [(q \vee p')(s \vee r')] \mid (p \vee r) \\
(q|p)' &= (q'|p)
\end{aligned}
\tag{10}
$$

Proof of Theorem 2: The result follows by using the equations 4, 5 and 9 to express $(q|p)(\omega)$ and $(s|r)(\omega)$ in terms of the original unconditioned propositions q, p, s, & r, and then collecting cases. For instance, the formula for disjunction goes as follows:

$$[(q|p) \vee (s|r)](\omega) = (q|p)(\omega) \vee (s|r)(\omega) \tag{11}$$

$$= \begin{cases} (q|p)(\omega) \vee (s|r)(\omega), & \text{if } (q|p)(\omega) \text{ and } (s|r)(\omega) \text{ are defined,} \\ (q|p)(\omega), & \text{if } (s|r)(\omega) \text{ is undefined,} \\ (s|r)(\omega), & \text{if } (q|p)(\omega) \text{ is undefined} \end{cases} \tag{12}$$

$$= \begin{cases} q(\omega) \vee s(\omega), & \text{if } p(\omega) = 1 \text{ and } r(\omega) = 1, \\ q(\omega), & \text{if } p(\omega) = 1 \text{ and } r(\omega) = 0, \\ s(\omega), & \text{if } p(\omega) = 0 \text{ and } r(\omega) = 1, \\ \text{undefined}, & \text{if } p(\omega) = 0 \text{ and } r(\omega) = 0 \end{cases} \tag{13}$$

$$= \begin{cases} (qp)(\omega) \vee (sr)(\omega), & \text{if } p(\omega) = 1 \text{ or } r(\omega) = 1, \\ \text{undefined}, & \text{if } p(\omega) = 0 \text{ and } r(\omega) = 0 \end{cases} \tag{14}$$

$$= \begin{cases} (qp \vee sr)(\omega), & \text{if } (p \vee r)(\omega) = 1, \\ \text{undefined}, & \text{if } (p \vee r)(\omega) = 0 \end{cases} \tag{15}$$

$$= [(qp \vee sr) \mid (p \vee r)](\omega). \tag{16}$$

Note, for instance, that if $(s|r)(\omega)$ is undefined, i.e. when $r(\omega) = 0$, then $(q|p)(\omega) \vee (s|r)(\omega) = (q|p)(\omega)$.

With the operations of "and" (juxtaposition or $\wedge$), "or" ($\vee$) and "not" (') the set L/L of ordered pairs $(q|p)$ of propositions includes an isomorphic copy of the original Boolean algebra of propositions according to the identification

$$(p|1) \rightarrow p, \tag{17}$$

and for any fixed non-zero proposition p, the conditionals $\{(q|p)$: all $q \in L\}$ form a Boolean algebra, which will be denoted $\mathcal{L}/p$. But it is not true that as a whole L/L together with these three operations forms a Boolean algebra. More on this later.

While the above formula for the disjunction ($\vee$) of two conditionals is given in reduced form, the formulas for the other two operations are not. In reduced form these other two become

$$\begin{aligned} (q|p)(s|r) &= (qpr' \vee p'sr \vee qpsr) \mid (p \vee r) \qquad (18) \\ (q|p)' &= (q'p \mid p) \end{aligned}$$

De Morgan's Laws for Conditionals: It is interesting to note that De Morgan's formulas have a counterpart here:

Theorem 3:
$$\begin{aligned} [(q|p) \vee (s|r)]' &= (q|p)' \wedge (s|r)' \qquad (19) \\ [(q|p) \wedge (s|r)]' &= (q|p)' \vee (s|r)' \end{aligned}$$

Proof of Theorem 3: $[(q|p) \vee (s|r)]' = [(qp \vee sr) \mid (p \vee r)]' = (qp \vee sr)' \mid (p \vee r)$ $= (qp)'(sr)' \mid (p \vee r) = (q' \vee p')(s' \vee r') \mid (p \vee r) = (q'|p) \wedge (s'|r) = (q|p)' \wedge (s|r)'$ using that $(p')' = p$. That proves the first formula. Using the first formula the dual formula follows: $[(q|p) \wedge (s|r)]' = [(q'|p)' \wedge (s'|r)']' = [[(q'|p) \vee (s'|r)]']' =$ $(q'|p) \vee (s'|r) = (q|p)' \vee (s|r)'$.

With respect to priority of operations, when parentheses are omitted, negation (') takes precedence and then conjunction (juxtaposition or $\wedge$) and then disjunction ($\vee$) and then conditioning (|). Thus $(sr \mid q \vee p')$ means $(sr) \mid (q \vee (p'))$.

The Conditional Closure: To obtain closure of operations in L/L, the conditioning process must be extended to the ordered pairs themselves --- to conditional conditionals. These are of the form $(q|p) \mid (s|r)$. [Those of the mixed forms, $((q|p) \mid s)$ and $(q \mid (s|r))$ for propositions q, p, s, and r, can be expressed as $(q|p) \mid (s|1)$ and $(q|1) \mid (s|r)$ respectively.]

Definition 6: For arbitrary conditionals $(q|p)$ and $(s|r)$, define

$$[(q|p) \mid (s|r)](\omega) = (q|p)(\omega) \mid (s|r)(\omega) \tag{20}$$

The following result reduces a conditional conditional to a single conditional of the original propositions.

Theorem 4: For arbitrary conditionals $(q|p)$ and $(s|r)$

$$(q|p) \mid (s|r) = q \mid (p\,(s \vee r')). \tag{21}$$

Proof of Theorem 4: As with the proof of the other operations above, the result follows by using the definition of $(q|p)(\omega)$ to express $(q|p)(\omega)$ and $(s|r)(\omega)$ in terms of the original unconditioned propositions q, p, s, & r, and then collecting and rephrasing the cases:

$$[(q|p) \mid (s|r)](\omega) = \begin{cases} (q|p)(\omega), & \text{if } (s|r)(\omega) \neq 0, \\ \text{undefined}, & \text{if } (s|r)(\omega) = 0 \end{cases} \tag{22}$$

$$= \begin{cases} (q|p)(\omega), & \text{if } (s|r)(\omega) = 1, \\ (q|p)(\omega), & \text{if } (s|r)(\omega) \text{ is undefined}, \\ \text{undefined}, & \text{if } (s|r)(\omega) = 0 \end{cases} \tag{23}$$

$$= \begin{cases} (q|p)(\omega), & \text{if } s(\omega) = 1 \text{ and } r(\omega) = 1, \\ (q|p)(\omega), & \text{if } r(\omega) = 0, \\ \text{undefined}, & \text{if } s(\omega) = 0 \text{ and } r(\omega) = 1 \end{cases} \tag{24}$$

$$= \begin{cases} (q|p)(\omega), & \text{if } s(\omega) \vee r'(\omega) = 1, \\ \text{undefined}, & \text{if } s(\omega) \vee r'(\omega) = 0 \end{cases} \tag{25}$$

$$= \begin{cases} (q|p)(\omega), & \text{if } (s \vee r')(\omega) = 1, \\ \text{undefined}, & \text{if } (s \vee r')(\omega) = 0 \end{cases} \tag{26}$$

$$= \begin{cases} q(\omega), & \text{if } [p(s \vee r')](\omega) = 1, \\ \text{undefined}, & \text{if } [p(s \vee r')](\omega) = 0 \end{cases} \tag{27}$$

$$= [q \mid p(s \vee r')](\omega). \tag{28}$$

Note that when the premise conditional (s|r) is undefined (i.e. r = 0) it has no effect on the conclusion conditional (q|p).

Corollary to Theorem 4: Applying the above theorem to the mixed form cases mentioned above yields:

$$((q|p) \mid s) = (q \mid ps) \tag{29}$$
$$(q \mid (s|r)) = (q \mid (s \vee r'))$$

Note that as a condition (s|r) is equivalent to $(q \vee p')$.

The collection L/L of all conditional propositions under the four operations "and" (juxtaposition or $\wedge$), "or" ($\vee$), "not" (') and "given" (|) forms a closed system that the author has called the *conditional closure* of the Boolean logic $\mathcal{L}$. The *conditional closure* will formally be denoted $\mathcal{L}/\mathcal{L}$.

Since $\mathcal{L}/\mathcal{L}$ is closed, compound conditional expressions can be reduced to simple conditionals of Boolean functions, which have well-defined conditional probabilities.

Truth Value Representation: The above development can be expressed in terms of 3-valued truth tables for conditional propositions (or restricted indicator functions). That is, for any fixed $\omega \in \Omega$, a conditional proposition is either true (1), or false (0) or undefined (U). This contrasts from the 2-valued propositions that arise from the characteristic functions of the various events in $\mathfrak{B}$. These 2-valued propositions are either true in ω or false in ω and always defined in ω.

Note that a particular conditional proposition may be true in some ω, false in other ω and undefined in still other ω. It is not accurate (except in a categorical Boolean algebra) to say without regard to a particular ω that each conditional proposition (q|p) is either true, false or undefined.

Note also that the third truth value is designated U for "undefined" not U for "uncertain". Perhaps this value is best expressed as "inapplicable". It is not a value between 0 and 1; it is a completely separate value.

The truth tables for the operations on conditional propositions (q|p) and (s|r) easily follow by considering all possible assignments of T, F, and U (1, 0 and Undefined) to the initial propositions and then applying the operations on conditionals:

		(s|r)(ω)		
	AND	T	F	U
	T	T	F	T
(q|p)(ω)	F	F	F	F
	U	T	F	U

		(s|r)(ω)		
	OR	T	F	U
	T	T	T	T
(q|p)(ω)	F	T	F	F
	U	T	F	U

NOT	T	F	U
	F	T	U

		(s|r)(ω)		
	GIVEN	T	F	U
	T	T	U	T
(q|p)(ω)	F	F	U	F
	U	U	U	U

It can be seen that with an appropriate extension of the functions max and min to include the additional domain value U, "or" corresponds to max, and "and" corresponds to min.

Algebraic Properties of $\mathcal{L}/\mathcal{L}$: While $\mathcal{L}/\mathcal{L}$ contains many Boolean algebras, it is not itself a Boolean algebra. In particular, $\mathcal{L}/\mathcal{L}$ is not wholly distributive. Nor does it in general have absolute complements. Furthermore there are no absolute units except the wholly undefined conditional (1|0). For instance, $(q|p) \wedge 1 = q \vee p' \neq (q|p)$ unless $p = 1$. For an elaboration on these matters see [3], p. 226-7.

Another interesting algebraic consequence of the conditional closure operations in $\mathcal{L}/\mathcal{L}$ is "non-monotonicity" [34]. Unlike the situation in a Boolean algebra, it is not true that the disjunction of two statements is necessarily entailed by each of the two component statements. If this seems strange consider the following example of a compound conditional: "If the store has Pepsi then buy Pepsi or if the store doesn't have Pepsi then buy Coke" The intent of the compound statement is to impose a double constraint. That is, if the store has Pepsi then I need only buy Pepsi for the compound conditional to be true. But if the store

doesn't have Pepsi and does have Coke then the Coke must be bought to satisfy the compound conditional. Without the conditional about Coke I would only need to purchase Pepsi if the store has it for the statement to be true, nothing being said in case of no Pepsi at the store. Thus the disjunction of two conditionals (with mutually inconsistent premises) is false when I fail to satisfy each separate conditional when it applies.

It turns out that when the premises of two conditionals are disjoint it doesn't matter whether the conditionals are disjoined or conjoined! The result is the same in English and in the conditional closure algebra. More on this later.

Probability in $\mathcal{L}/\mathcal{L}$: Since complex conditional expressions can be reduced in $\mathcal{L}/\mathcal{L}$ to a single conditional of Boolean expressions, they all have an implied conditional probability. For the disjunction of any two conditional propositions (q|p) and (s|r), the probability can be determined according to the following formula:

Theorem 5: For any conditional propositions (q|p) and (s|r), $P(p) \neq 0 \neq P(r)$,

$$P((q|p) \vee (s|r)) = P(p \mid p \vee r)\, P(q|p) + P(r \mid p \vee r)\, P(s|r) - P(qpsr \mid p \vee r)$$

Note that for p = r, this reduces to the ordinary probability of a disjunction.

Proof of Theorem 5: $P((q|p) \vee (s|r)) = P(qp \vee sr \mid p \vee r)$

$$\begin{aligned}
&= P(qp \mid p \vee r) + P(sr \mid p \vee r) - P(qpsr \mid p \vee r) \\
&= P((q|p) \mid p \vee r)\, P(p \mid p \vee r) + P((s|r) \mid p \vee r)\, P(r \mid p \vee r) - P(qpsr \mid p \vee r) \\
&= P(q \mid p(p \vee r))\, P(p \mid p \vee r) + P(s \mid r(p \vee r))\, P(r \mid p \vee r) - P(qpsr \mid p \vee r) \\
&= P(q|p)\, P(p \mid p \vee r) + P(s|r)\, P(r| p \vee r) - P(qpsr \mid p \vee r).
\end{aligned}$$

There is also a non-trivial formula for $P((q|p) \wedge (s|r))$:

Theorem 6: Under the hypothesis of the preceding theorem,

$$P((q|p) \wedge (s|r)) = P(p \mid p \vee r)\, P(qr'|p) + P(r \mid p \vee r)\, P(sp'|r) + P(qpsr \mid p \vee r).$$

Proof of Theorem 6: $P((q|p) \wedge (s|r)) = P(qpr' \vee srp' \vee qpsr \mid p \vee r)$

$$\begin{aligned}
&= P(qpr' \mid p \vee r) + P(srp' \mid p \vee r) + P(qpsr \mid p \vee r) \\
&= P(p \mid p \vee r)\, P((qr'|p)|(p \vee r)) + P(r \mid p \vee r)\, P((sp'|r)|(p \vee r)) + P(qpsr \mid p \vee r) \\
&= P(p \mid p \vee r)\, P(qr'| p(p \vee r)) + P(r \mid p \vee r) P(sp'| r(p \vee r)) + P(qpsr| p \vee r) \\
&= P(p \mid p \vee r)\, P(qr'|p) + P(r \mid p \vee r)\, P(sp'|r) + P(qpsr \mid p \vee r).
\end{aligned}$$

Note that the last term of the formulas of the last two theorems, namely $P(qpsr \mid p \vee r)$, can be expressed as $P(pr \mid p \vee r)\, P(qs \mid pr)$.

In view of the very large sample spaces associated with even a small number of variables, it is not practical to attempt to enumerate possibilities and calculate probabilities and conditional probabilities from scratch. Formulas like those of Theorems 5 and 6 allowing local calculation of conditional probabilities via partitions are essential for practical determination of conditional probabilities in artificial intelligence applications.

3. Varieties of Deduction

Deduction in $\mathcal{L}$. In a Boolean algebra $\mathcal{L}$, one proposition may (necessarily) imply a second proposition. In the simplest case the truth of a proposition such as $(p \vee 1)$ is implied by the Boolean algebra axioms, which of course, are true in all models ω of $\mathcal{L}$. For instance the truth of $(p \vee 1)$ follows from the laws of complements, associativity and idempotency: $p \vee 1 = p \vee (p \vee p') = (p \vee p) \vee p' = p \vee p' = 1$. In terms of indicator functions, $(p \vee 1)(\omega) = 1$ for all $\omega \in \Omega$. That is, $(p \vee 1)$ is the unity function on Ω. The truth of $(p \vee 1)$ can also be expressed using the following familiar partial ordering on $\mathcal{L}$.

Definition 7: $p \leq q$ if and only if $pq = p$.

With this definition it is easy to show the following equivalent form:

$$p \leq q \text{ if and only if } p \vee q = q. \tag{30}$$

If $p \leq q$ holds, we say p entails (necessarily implies) q. In terms of $\leq$ the fact that $(p \vee 1)$ is always true (= 1) can be expressed as $1 \leq (p \vee 1)$ and $(p \vee 1) \leq 1$. In this way axioms and theorems of the form $p = q$ can all be expressed in terms of $\leq$:

$$p = q \text{ if and only if } p \leq q \text{ and } q \leq p. \tag{31}$$

In terms of indicator functions, $(p \leq q)$ is just functional inequality in the 2-element Boolean algebra. That is, $(p \leq q)$ means $p(\omega) \leq q(\omega)$, for all $\omega \in \Omega$, where $0 \leq 1$. Note that if $p \leq q$ then easily $q' \leq p'$ and for any proposition r, $pr \leq qr$ and $(p \vee r) \leq (q \vee r)$. The converses are also true. In addition $p \leq q$ if and only if $pq' = 0$.

Besides the two equivalent forms given above expressing "p entails q", there are at least two other ways to express $p \leq q$:

$$p \leq q \text{ if and only if } q \vee p' = 1 \tag{32}$$

and

$$p \leq q \text{ if and only if } (q|p) = (1|p) \tag{33}$$

The first statement follows because if $p \leq q$ then $q = p \vee q$. So $q \vee p' = (p \vee q) \vee p' = 1$. Conversely, if $q \vee p' = 1$ then $(q \vee p')p = (1)(p)$. So $qp \vee p'p = p$. Thus $qp = p$. So $p \leq q$.

The second statement follows because if $p \le q$ then $pq = p$. So $(q|p) = (qp|p) = (p|p) = (1|p)$. Conversely, if $(q|p) = (1|p)$ then $qp = (p)(1)$. That is, $qp = p$. So $q \le p$.

According to the definition of equivalent conditionals, the proposition q is equivalent to p in the Boolean algebra $\mathcal{L}/p$ if and only if $(q|p) = (p|p)$, that is, if and only if $qp = p$, which is just $p \le q$. Thus, in $\mathcal{L}/p$, q is equivalent to p if and only if q is entailed by p. We say that q is in the equivalence class generated by p. In other terminology, q is said to be in the *filter class* generated by p or in the *sum ideal* generated by p. This is the equivalence class $\langle p|p \rangle = \{q \in L: p \le q\}$. Thus, still another way to express "p entails q" is to say that q is in the sum ideal generated by p.

Deduction in $\mathcal{L}/\mathcal{L}$: Unlike the situation existing in $\mathcal{L}$ it is not true that the following two potential definitions for entailment are equivalent in $\mathcal{L}/\mathcal{L}$:

Definition 8 (Conjunctive Implication): $(q|p)$ conjunctively implies $(s|r)$, denoted $(q|p) \le_\wedge (s|r)$, if and only if $(q|p) \wedge (s|r) = (q|p)$. That is,

$$(q|p) \le_\wedge (s|r) \quad \text{if and only if} \quad (q|p) \wedge (s|r) = (q|p)$$

Definition 9 (Disjunctive Implication): $(q|p)$ disjunctively implies $(s|r)$, denoted $(q|p) \le_\vee (s|r)$, if and only if $(q|p) \vee (s|r) = (s|r)$. That is,

$$(q|p) \le_\vee (s|r) \quad \text{if and only if} \quad (q|p) \vee (s|r) = (s|r)$$

In fact if the properties of both definitions hold then, as will be shown below, $p = r$ and $qp \le sr$. Thus the situation reduces to the special case of equal premises, i.e., $q \le s$ on the subset on which p (and r) are true.

In this restricted context of *equivalent premises*, p and r, there are various equivalent ways to express "$(q|p)$ entails $(s|p)$" including: $(q|p) \le (s|p)$, $(q|p) \le_\vee (s|p)$, $(q|p)' \vee (s|p) = (1|p)$, and $(s|p)|(q|p) = 1|(q|p)$. These are all equivalent to the statement $qp \le sp$. Yet another way to express this is $(s|p) \in \langle q|p \rangle$, that $(s|p)$ is in the filter class (sum ideal) generated by $(q|p)$. Each of these facts follow in a few steps. Yet it turns out that none of the first four relations are equivalent when the premises of the conditionals are not equivalent!

While $\le_\wedge$ and $\le_\vee$ are not equivalent in $\mathcal{L}/\mathcal{L}$, they do both constitute partial orderings:

Theorem 7: Both $\le_\wedge$ and $\le_\vee$ establish partial orderings on $\mathcal{L}/\mathcal{L}$.

Proof of Theorem 7: The proof follows by routine application of the operations to show reflexivity, antisymmetry and transitivity. For example, reflexivity of $\leq_\wedge$ follows from $(q|p) \wedge (q|p) = [(q \vee p')(q \vee p')] \mid (p \vee p) = [(q \vee p') \mid p] = [(qp \vee p'p) \mid p] = [(qp \vee 0) \mid p] = (qp|p) = (q|p)$.

The following two theorems express the inequalities $(q|p) \leq_\wedge (s|r)$ and $(q|p) \leq_\vee (s|r)$ in terms of the partial ordering on the original propositions.

Theorem 8: $(q|p) \leq_\wedge (s|r)$ if and only if $r \leq p$ and $qp \leq sr \vee r'$.

Corollary to Theorem 8: $(q|p) \leq_\wedge (s|r)$ if and only if $r \leq p$ and $qr \leq sr$.

Theorem 9: $(q|p) \leq_\vee (s|r)$ if and only if $p \leq r$ and $qp \leq sr$.

Corollary to Theorem 9: $(q|p) \leq_\vee (s|r)$ if and only if $p \leq r$ and $qp \leq sp$

Proof of Theorem 8: Suppose $(q|p) \leq_\wedge (s|r)$. Therefore $(q|p)(s|r) = (q|p)$. Then $(qpr' \vee p'sr \vee qpsr \mid p \vee r) = (q|p) = (qp|p)$. So $p \vee r = p$ and $qpr' \vee p'sr \vee qpsr = qp$. From $p \vee r = p$ it follows that $r \leq p$. So $rp' = 0$. Combining these results yields $qp = qpr' \vee p'sr \vee qpsr = qpr' \vee 0 \vee qps = qp(r' \vee s) = qp(r' \vee sr)$. Thus $qp \leq (sr \vee r')$. Conversely, if $(r \leq p)$ and $(qp \leq sr \vee r')$, then $qp = qp(r' \vee sr) = qpr' \vee qpsr = qpr' \vee 0 \vee qpsr = qpr' \vee p'sr \vee qpsr$. Therefore $(q|p) = (qp|p) = (qp \mid p \vee r) = (qpr' \vee p'sr \vee qpsr \mid p \vee r) = (q|p)(s|r)$.

To prove the Corollary to Theorem 8, note that if $(r \leq p)$ and $(qp \leq sr \vee r')$, then $qpr \leq (sr \vee r')r$. So $qr \leq sr$. Conversely, if $(r \leq p)$ and $(qr \leq sr)$, then $qrp \leq srp = sr$. So $qp = qpr \vee qpr' \leq sr \vee qpr' \leq sr \vee r'$.

Proof of Theorem 9: Suppose $(q|p) \leq_\vee (s|r)$. That is, $(q|p) \vee (s|r) = (s|r)$. Therefore $(qp \vee sr \mid p \vee r) = (s|r) = (sr|r)$. So, $p \vee r = r$ and $qp \vee sr = sr$. So $p \leq r$ and $qp \leq sr$. Conversely, if $p \leq r$ and $qp \leq sr$, then $p \vee r = r$ and $qp \vee sr = sr$. So $(s|r) = (sr|r) = (qp \vee sr \mid p \vee r) = (q|p) \vee (s|r)$. That is, $(q|p) \leq_\vee (s|r)$.

To prove the corollary to Theorem 9 note that if $p \leq r$ and $qp \leq sr$ then $qpp \leq srp = sp$. So $qp \leq sp$. Conversely, if $qp \leq sp$ and $p \leq r$ then $qp \leq sp \leq sr$.

Note that as a consequence of Theorem 8 it turns out that $(q \vee p') \leq_\wedge (q|p)$ since $p \leq 1$ and $(q \vee p')p \leq qp$.

Non-Monotonicity of Deduction: As appealing as this definition of entailment seems, it nevertheless appears at first to have a serious flaw, namely: If $(q|p) \leq_\wedge (s|r)$ then it

does not follow that $P(q|p) \leq P(s|r)$. This issue was brought to the attention of the author by H. T. Nguyen and it also appears in Dubois and Prade [20], both suggesting that a deduction relation is inappropriate as an entailment relation in $\mathcal{L}/\mathcal{L}$ unless it is monotonic in the sense that if $(q|p)$ entails $(s|r)$ then $P(q|p)$ should not be greater than $P(s|r)$. But $\leq_\wedge$ does not satisfy this relation:

For example, let $p = 1$, $P(q) = 1/2$, $r = q'$ and $s = q$. Then easily $(q|p) \leq_\wedge (s|r)$ but $P(q|p) = 1/2$ while $P(s|r) = 0$. In fact, $(q \vee p') \leq_\wedge (q|p)$ but it was shown earlier that $P(q|p) \leq P(q \vee p')$, not the other way around. Nevertheless, in thinking about conditional logic and probability one must be flexible with one's conclusions. One must be "non-monotonic" in one's thinking! It might at first seem obvious that $(q|p) \leq_\wedge (s|r)$ should imply $P(q|p) \leq P(s|r)$. The idea comes swiftly to the mind and just as quickly to the tongue, but is it really so reasonable as a general rule?

Upon second thought, it seems to the author to be rather questionable whether one should insist that whenever one conditional (with its own premise and probability of application) entails a second conditional (with its own premise and probability of application) then their conditional probabilities must be so ordered. Rather, in the propagation of conditional probabilities through a network of logically related conditional propositions, one must perhaps allow for increasing *or decreasing* conditional probabilities. A similar observation can be made for $\leq_\vee$. For instance let $0 < P(q) < 1$. Then $(q|q) \leq_\vee (q|1)$, i.e., $(q|q)$ disjunctively implies $(q|1)$ because $q \leq 1$ and $qq \leq (q)1$. However $P(q|1) = P(q) < 1 = P(q|q)$.

In its initial formulation *non-monotonicity* [34] arises from the observation in probability theory that $P(q \mid p \wedge r)$ can be less, more or equal to $P(q|p)$ even though $p \wedge r$ entails p. The lack of monotonicity of $\leq_\wedge$ is also well exhibited by considering the two forms $(q|p)$ and $(q \vee p')$. As shown earlier $P(q|p) \leq P(q \vee p')$, but it is also true that $(q \vee p') \leq_\wedge (q|p)$. On the other hand it is also easy to show that $(q|p) \leq_\vee (q \vee p')$.

Nevertheless, if $(q|p) \leq_\wedge (s|r)$ then it has been shown that $qp \leq sr \vee r'$. That is, whenever $(q|p)$ is true then either $(s|r)$ is true or else r is false. It will later be shown that if $(q|p) \leq_\wedge (s|r)$ then $(q \vee p') \leq (s \vee r')$ and so $P(q|p) \leq P(q \vee p') \leq P(sr \vee r')$. That is, if $(q|p) \leq_\wedge (s|r)$ then $P(q|p) \leq P(sr \vee r')$.

On the other hand, it is not even true in general that if $(q|p) \leq_\vee (s|r)$ then $P(q|p) \leq P(sr \vee r')$. For instance, let $0 \neq q = p \leq s \neq 1 = r$. So $(q|p) \leq_\vee (s|r)$. However $P(q|p) = 1 > P(sr \vee r') = P(s)$.

In addition to $\leq_\wedge$ and $\leq_\vee$ there are at least two other candidates for expressing the notion that an arbitrary conditional $(q|p)$ entails a second arbitrary conditional $(s|r)$. The following two definitions formalize these relations:

Definition 10 (Conditional Implication): $(q|p)$ conditionally implies $(s|r)$, denoted $(q|p) \leq_c (s|r)$, if and only if $(s|r)$ is true given $(q|p)$. That is,

$$(q|p) \leq_c (s|r) \quad \text{if and only if} \quad (s|r)|(q|p) = (1|r)|(q|p).$$

Definition 11 (Material Implication): $(q|p)$ materially implies $(s|r)$, denoted $(q|p) \leq_m (s|r)$, if and only if either $(s|r)$ is true or $(q|p)$ is false. That is,

$$(q|p) \leq_m (s|r) \quad \text{if and only if} \quad (s|r) \vee (q|p)' = (1 \mid r \vee p).$$

The following theorems express these equations in terms of the partial ordering $\leq$ on the original propositions.

Theorem 10: $(q|p)$ conditionally implies $(s|r)$, that is, $(s|r)|(q|p) = (1|r)|(q|p)$ if and only if any one of the following hold:

$$r(q \vee p') \leq s,$$
$$(q \vee p') \leq (s \vee r'),$$
$$rs' \leq pq'.$$

This theorem gives three equivalent ways of expressing the equation $(s|r)|(q|p) = (1|r)|(q|p)$, that $(q|p)$ conditionally implies $(s|r)$: $r(q \vee p') \leq s$ means that if the premise r of the conclusion is true and the premise conditional, $(q|p)$, is not false then the conclusion s is true. $q \vee p' \leq s \vee r'$ means if the premise conditional is not false then the conclusion conditional is not false. $rs' \leq pq'$ means that if the conclusion conditional is false then the premise conditional is false.)

Proof of Theorem 10: Suppose $(q|p) \leq_c (s|r)$. Then $(s|r)|(q|p) = [s \mid r(q \vee p')]$ and $(1|r)|(q|p) = [1 \mid r(q \vee p')]$. So $sr(q \vee p') = r(q \vee p')$. Therefore $r(q \vee p') \leq s$. By reversing the steps the converse follows. The 2nd and 3rd relations of Theorem 10 follow since $r(q \vee p') \leq s$ if and only if $r(q \vee p')s' = 0$ if and only if $rs' \leq (q \vee p')' = pq'$. In addition, $r(q \vee p')s' = 0$ if and only if $(q \vee p') \leq (rs')' = (s \vee r')$.

Here again conditional implication ($\leq_c$) is non-monotonic in the sense that $(q|p) \leq_c (s|r)$ does not imply $P(q|p) \leq P(s|r)$. For instance, let $p = 1$, $s = q$, and $r = q'$ with $0 < P(q) < 1$. Then $(q|p) \leq_c (s|r)$ since $q \vee p' = q$ and $s \vee r' = q$. But $P(q|p) = P(q) > 0 = P(s|r)$.

For $\leq_c$ it is true that both $(q|p) \leq_c (q \vee p')$ and $(q \vee p') \leq_c (q|p)$ but clearly, $(q|p) \neq (q \vee p')$. Two propositions may conditionally imply each other without being equivalent. Nevertheless, as is so for both $\leq_\wedge$ and $\leq_m$, if $(q|p) \leq_c (s|r)$ then $P(q|p) \leq P(sr \vee r')$. That is, if $(q|p) \leq_c (s|r)$ then the probability that $(q|p)$ is true is less than or equal to the probability that $(s|r)$ is true or undefined.

The following theorem and its corollaries give three equivalent ways to express that $(q|p)$ materially implies $(s|r)$, i.e., that $(q|p) \leq_m (s|r)$.

Theorem 11: $(s|r) \vee (q|p)' = (1 \mid r \vee p)$ if and only if

$$qp \leq sr \text{ and } r(q \vee p') \leq s.$$

Proof of Theorem 11: $(s|r) \vee (q|p)' = (1 \mid r \vee p)$ if and only if $(sr \vee q'p \mid r \vee p) = (1 \mid r \vee p)$ if and only if $sr \vee q'p = r \vee p$. Multiplying the last equation by qp yields $srqp = rqp \vee qp = qp$. So $qp \leq sr$. Furthermore, multiplying the equation $sr \vee q'p = r \vee p$ by $s'r$ yields $0 \vee q'ps'r = s'r \vee ps'r = s'r$. So $s'r \leq q'p$. So $s'r(q'p)' = 0$. So $s'r(q \vee p') = 0$. Therefore $r(q \vee p') \leq s$. That completes one direction of the proof. Conversely, if $qp \leq sr$ and $r(q \vee p') \leq s$, then $s'r(q \vee p') = 0$. That is, $s'r \leq (q \vee p')' = q'p$. So $qp \leq sr$ and $s'r \leq q'p$. therefore, $sr \vee q'p = (sr \vee qp) \vee (q'p \vee s'r) = (qp \vee q'p) \vee (sr \vee s'r) = p \vee r$. So $sr \vee q'p = r \vee p$. This completes the proof of Theorem 11.

As before the second inequality can be expressed in several ways:

Corollary to Theorem 11: $(s|r) \vee (q|p)' = (1 \mid r \vee p)$ if and only if either of the following hold:

$$qp \leq sr \text{ and } (q \vee p') \leq (s \vee r'),$$
$$qp \leq sr \text{ and } rs' \leq pq'.$$

Clearly from the last two theorems, the statement "(s|r) is true given (q|p) is true" (conditional implication) is weaker than the statement that "either (s|r) is true or (q|p) is false" (material implication).

It is important here as before to determine whether $(q|p) \leq_m (s|r)$ implies $P(q|p) \leq P(s|r)$. It was I. R. Goodman [22] who first proved this result as well as its more difficult converse; H. Prade & D Dubois [20] have also made this observation. The following proof is offered without the requirement of atomicity of the Boolean Algebra $\mathcal{L}$.

Theorem 12: Let a, b, c, and d be four propositions of $\mathcal{L}$. If $P(a|b) \leq P(c|d)$ for every probability measure P on $\mathcal{L}$ for which $P(b) \neq 0 \neq P(d)$ then either $(a|b) \leq_m (c|d)$ or

(ab = 0) or (d $\leq$ c). Conversely, if either (a|b) $\leq_m$ (c|d) or (ab = 0) or (d $\leq$ c) in $\mathcal{L}/\mathcal{L}$ then P(a|b) $\leq$ P(c|d).

Proof of Theorem 12: Prove the converse first: If (ab = 0) or (d $\leq$ c) then easily P(a|b) $\leq$ P(c|d) since either P(a|b) = 0 or P(c|d) = 1. Otherwise, ab $\leq$ cd and c'd $\leq$ a'b. Then P(a|b) = P(ab)/P(b) = P(ab)/[P(ab) + P(a'b)] = 1/[1 + P(a'b)/P(ab)] $\leq$ 1/[1 + P(c'd)/P(cd)] = P(cd)/[P(cd) + P(c'd)] = P(c|d), which proves the converse. Now suppose that neither (ab = 0) nor (d $\leq$ c) is true. So ab $\neq$ 0 and c'd $\neq$ 0. Now in any case

$$ab = abcd \vee abc'd \vee abcd' \vee abc'd \tag{34}$$

and

$$c'd = abc'd \vee a'bc'd \vee ab'c'd \vee a'b'c'd \tag{35}$$

Note that in these expansions, only (abc'd) is common to both. The conjunction of any other pair of propositions is 0. Thus any assignment of non-negative probabilities to these seven propositions whose sum is 1 determines a probability measure P on the subalgebra generated by a,b,c and d for which P(b) $\neq$ 0 $\neq$ P(d). It will be shown that all but abcd and a'bc'd must be 0 and that the latter must not be 0. Suppose first that the common proposition abc'd $\neq$ 0. Then assign it probability weight 1. In this case P(b) $\geq$ P(ab) $\geq$ P(abc'd) = 1 but P(cd) = P(d) - P(c'd) = 1 - 1 = 0. So P(a|b) = 1 but P(c|d) = 0, which is a contradiction. Next suppose either abcd' or abc'd' is not 0. In this case assign it probability weight 1/2 and assign c'd probability 1/2. Therefore P(ab) $\geq$ 1/2 but P(cd) = P(d) - P(c'd) = 1/2 - 1/2 = 0. So P(a|b) $\geq$ 1/2 but P(c|d) = 0, which is again a contradiction. Therefore 0 $\neq$ ab = abcd. So ab $\leq$ cd. Next suppose either ab'c'd or a'b'c'd is not 0. In that case assign it probability weight 1/2 and assign abcd probability 1/2. Then P(ab) = 1/2 and P(b) = 1/2 but P(cd) = P(d) - P(c'd) = 1 - 1/2 = 1/2. So P(a|b) = 1 but P(c|d) = 1/2, again a contradiction. Therefore 0 $\neq$ c'd = a'bc'd. So c'd $\leq$ a'b. This completes the proof.

In view of the preceding, since (p)(q|p) $\leq$ (r)(s|r) is equivalent to pq $\leq$ sr, having the latter relation together with (q|p) $\leq_c$ (s|r), which is equivalent to s'r $\leq$ q'p, yields that P(q|p) $\leq$ P(s|r). If instead of (q|p) $\leq_c$ (s|r), one has (q|p) $\leq_\vee$ (s|r), then again one has (q|p) $\leq_m$ (s|r), and so again P(q|p) $\leq$ P(s|r) as well as p $\leq$ r.

The following two theorems and the corollary relate conjunctive and conditional implication:

Theorem 13: If (q|p) $\leq_\wedge$ (s|r) then (q $\vee$ p') $\leq$ (s $\vee$ r'). That is, conjunctive implication implies conditional implication.

Corollary to Theorem 13: If $(q|p) \leq_{\wedge} (s|r)$ then $P(q|p) \leq P(q \vee p') \leq P(s \vee r')$.

Theorem 14: If $(q \vee p') \leq (s \vee r')$ and $r \leq p$ then $(q|p) \leq_{\wedge} (s|r)$. That is, with $(r \leq p)$ conditional implication implies conjunctive implication.

Proofs of Theorems 13 and 14: If conjunctive implication holds then $r \leq p$ and $qr \leq sr$. So $p' \leq r'$. So $q \vee p' = qp \vee p' \leq sr \vee p' \leq sr \vee r' = s \vee r'$. Therefore, $q \vee p' \leq s \vee r'$, which completes the proof of Theorem 13. The corollary is obvious. For Theorem 14, if $(q \vee p') \leq (s \vee r')$ and $r \leq p$ then $qp \leq qp \vee p' = q \vee p' \leq s \vee r' = sr \vee r'$. So $qp \leq sr \vee r'$. Therefore $(q|p) \leq_{\wedge} (s|r)$ by Theorem 8.

While conditional implication is weaker than both conjunctive implication and material implication, conditional implication is not weaker than disjunctive implication. Nor is disjunctive implication weaker than conditional implication. Concerning the relationship between disjunctive and conditional implication there are the following theorems:

Theorem 15: If $(q|p) \leq_{c} (s|r)$ and $p \leq r$ then $(q|p) \leq_{\vee} (s|r)$.

Theorem 16: If $(q|p) \leq_{\vee} (s|r)$ and $p' \leq s \vee r'$ then $(q|p) \leq_{c} (s|r)$.

Proofs of Theorems 15 and 16: For Theorem 15, suppose $(q \vee p') \leq (s \vee r')$ and $p \leq r$. So $(q \vee p')p \leq s \vee r')p$. That is, $qp \leq sp \vee r'p = sp \vee 0 = sp$. So $qp \leq sp$. Therefore $(q|p) \leq_{\vee} (s|r)$. For Theorem 16, suppose $(q|p) \leq_{\vee} (s|r)$ and $p' \leq s \vee r'$. Therefore $q \vee p' = qp \vee p' \leq sp \vee p' \leq s \vee p' = s \vee (s \vee r') = s \vee r'$. That is, $q \vee p' \leq s \vee r'$. So $(q|p) \leq_{c} (s|r)$. This completes the proof of Theorem 16.

Entailment in $\mathcal{L}/\mathcal{L}$: And so it seems that there are different entailments for different situations and most do not impose monotonicity of conditional probability.

From the preceding results it is easy to see the following relationships:

If $(q|p) \leq_{m} (s|r)$ and $r \leq p$ then $(q|p) \leq_{\wedge} (s|r)$ (36)
If $(q|p) \leq_{\wedge} (s|r)$ and $qp \leq sr$ then $(q|p) \leq_{m} (s|r)$ and $r \leq p$

If $(q|p) \leq_{m} (s|r)$ and $p \leq r$ then $(q|p) \leq_{\vee} (s|r)$ (37)
If $(q|p) \leq_{\vee} (s|r)$ and $(q|p) \leq_{c} (s|r)$ then $q|p) \leq_{m} (s|r)$ and $p \leq r$

If $(q|p) \leq_{\wedge} (s|r)$ and $p \leq r$ then $(q|p) \leq_{\vee} (s|r)$ and $p = r$ (38)
If $(q|p) \leq_{\vee} (s|r)$ and $r \leq p$ then $(q|p) \leq_{\wedge} (s|r)$ and $p = r$

If $(q|p) \leq_{m} (s|r)$ then $(q|p) \leq_{c} (s|r)$ (39)
If $(q|p) \leq_{c} (s|r)$ and $qp \leq sr$ then $(q|p) \leq_{m} (s|r)$

It is interesting to note that $\leq_\wedge$, $\leq_\vee$ and $\leq_m$ generate lattices in $\mathbf{L}/\mathbf{L}$ but $\leq_c$ is not a partial ordering because it fails to be antisymmetric.

Nevertheless, $\leq_c$ is a quasi-ordering ([25], p. 4). Of course by equating all conditionals for which $\leq_c$ holds in both directions, a partial ordering arises. However this entails making (q|p) and (q $\vee$ p') equivalent, which is not desirable except when both are certain. The following theorem gives three ways to express the fact that two conditional propositions are equivalent.

Theorem 17: (q|p) = (s|r) if and only if any one of the following are true:

$$(q|p) \leq_\wedge (s|r) \text{ and } (s|r) \leq_\wedge (q|p),$$
$$(q|p) \leq_\vee (s|r) \text{ and } (s|r) \leq_\vee (q|p),$$
$$(q|p) \leq_m (s|r) \text{ and } (s|r) \leq_m (q|p).$$

Proof of Theorem 17: Clearly, if both $(q|p) \leq_\wedge (s|r)$ and $(s|r) \leq_\wedge (q|p)$ then $r \leq p$ and $qr \leq sr$, and $p \leq r$ and $sp \leq qp$. So $p = r$ and $qp \leq sr$, and $sr \leq qp$. So $p = r$ and $qp = sr$. So $(q|p) = (s|r)$. The converse is also easy. Similarly for $\leq_\vee$. Next, if both $(q|p)' \vee (s|r) = (1 \mid p \vee r)$ and $(s|r)' \vee (q|p) = (1 \mid p \vee r)$ then using the Corollary to Theorem 11, $qp \leq sr$, $sr \leq qp$, $s'r \leq q'p$ and $q'p \leq s'r$. So $qp = sr$ and $s'r = q'p$. So $r = sr \vee s'r = qp \vee q'p = p$. Therefore $(q|p) = (s|r)$. Conversely, if $(q|p) = (s|r)$, then $qp = sr$ and $p = r$. So both $qp \leq sr$ and $sr \leq qp$. Therefore, $s'r = s'(q \vee q')r = s'qr \vee s'q'r = s'qp \vee s'q'p = s'(sr) \vee s'q'p = 0 \vee s'q'p \leq q'p$. So $s'r \leq q'p$. By symmetry, $q'p \leq s'r$. So $s'r = q'p$. This completes the proof.

Theorem 18: The conditional closure $\mathbf{L}/\mathbf{L}$ is a lattice under the partial ordering $\leq_\wedge$, where the least upper bound (LUB) and greatest lower bound (GLB) are given by

$$\mathrm{LUB}_\wedge\,(q|p,s|r) = (q \vee s) \mid (pr) = (qp \vee sr) \mid (pr),$$

$$\mathrm{GLB}_\wedge\,(q|p,s|r) = [(q \vee p')(s \vee r')] \mid (p \vee r) = (q|p) \wedge (s|r).$$

Proof of Theorem 18: Firstly, $(q \vee s)|(pr)$ is an upper bound of (q|p) because $pr \leq p$ and $q(pr) \leq (q \vee s)pr$, using the Corollary to Theorem 8. By symmetry $(q \vee s)|(pr)$ is also an upper bound of (s|r). Now if (t|u) is any upper bound of both (q|p) and (s|r) then both $u \leq p$ and $u \leq r$. So $u \leq pr$. By the Corollary, both $qu \leq tu$ and $su \leq tu$. Therefore $(q \vee s)u = qu \vee su \leq tu \vee su \leq tu \vee tu = tu$. Thus $(q \vee s \mid u) \leq_\wedge (t|u)$. But $(q \vee s \mid pr) \leq_\wedge (q \vee s \mid u)$. So by transitivity of $\leq_\wedge$, $(q \vee s \mid pr) \leq_\wedge (t|u)$. Therefore $(q \vee s \mid pr)$ is the LUB of (q|p) and (s|r).

To show that $(q|p) \wedge (s|r) = [(q \vee p')(s \vee r')] \mid (p \vee r)$ is the GLB of $(q|p)$ and $(s|r)$, first note that both $p \le p \vee r$ and $r \le p \vee r$. Furthermore, $(q \vee p')(s \vee r')p = (qp \vee 0)(s \vee r') = qp(s \vee r') \le qp$. Similarly, $(q \vee p')(s \vee r')r = (q \vee p')(sr \vee 0) \le sr$. Therefore $(q \vee p')(s \vee r') \mid (p \vee r)$ is a lower bound for $(q|p)$ and $(s|r)$. Now if $(t|u)$ is any lower bound of $(q|p)$ and $(s|r)$ then both $p \le u$ and $r \le u$, and so $(p \vee r) \le u$. Furthermore, since both $tp \le qp$ and $tr \le sr$, then both $t \le q \vee p'$ and $t \le s \vee r'$. This follows from $t = tp \vee tp' \le qp \vee tp' \le qp \vee p' = q \vee p'$ and $t = tr \vee tr' \le sr \vee tr' \le sr \vee r' = s \vee r'$. Therefore, $t \le (q \vee p')(s \vee r')$. Thus $(t|u) \le_{\wedge} (t \mid p \vee r) \le_{\wedge} (q \vee p')(s \vee r') \mid (p \vee r)$ because $(p \vee r) \le u$ and $(p \vee r)t \le (p \vee r)(q \vee p')(s \vee r')$. This completes the proof of Theorem 18.

Theorem 19: $\mathcal{L}/\mathcal{L}$ is a lattice under the partial ordering $\le_{\vee}$ where the $\text{LUB}_{\vee}$ and $\text{GLB}_{\vee}$ are given by

$$\text{LUB}_{\vee}\ (q|p,s|r) = (q|p) \vee (s|r) = (qp \vee sr) \mid (p \vee r),$$

$$\text{GLB}_{\vee}\ (q|p,s|r) = (qs|pr).$$

Proof of Theorem 19: $(qs|pr)$ is a lower bound of $(q|p)$ and $(s|r)$ because $pr \le p$ and $pr \le r$, and because $(qs)(pr) \le qp$ and $(qs)(pr) \le sr$. Now if $(t|u)$ is any lower bound of $(q|p)$ and $(s|r)$, then $u \le p$, $u \le r$, $tu \le qp$ and $tu \le sr$. Therefore $(t|u) \le_{\vee} (qs|pr)$ because $u \le pr$ and $tu \le (qp)(sr)$. This completes the GLB part of the proof. $(qp \vee sr \mid p \vee r)$ is an upper bound of $(q|p)$ and $(s|r)$ because $p \le p \vee r$ and $r \le p \vee r$, and because $qp \le (qp \vee sr)(p \vee r) = qp \vee sr$, and similarly, $sr \le qp \vee sr$. Now if $(t|u)$ is any upper bound of $(q|p)$ and $(s|r)$, then $p \le u$, $r \le u$, $qp \le tu$ and $sr \le tu$. Therefore $(qp \vee sr \mid p \vee r) \le_{\vee} (t|u)$ because $p \vee r \le u$ and $(qp \vee sr)(p \vee r) = qp \vee sr \le tu$. This completes the proof.

While disjunctive implication establishes a full lattice in $\mathcal{L}/\mathcal{L}$, it doesn't appear to be universally appropriate for purposes of entailment. For instance consider the two conditionals $(p|p)$ and $(p|1)$ where $p \ne 1$. Then clearly $(p|p) \le_{\vee} (p|1)$. But $(p|p)$ is certain or undefined whereas $(p|1)$ is just p, and so is uncertain and possibly improbable but it is wholly defined. Nevertheless, $\le_{\vee}$ is appropriate in some circumstances.

Theorem 20: Material implication $(\le_m)$ establishes a partial ordering on $\mathcal{L}/\mathcal{L}$.

Proof of Theorem 20: $(q|p) \le_m (q|p)$ since $(q|p) \vee (q|p)' = (q|p) \vee (q'|p) = (1|p) = (1 \mid p \vee p)$. So $\le_m$ is reflexive. If $(q|p) \le_m (s|r)$ and $(s|r) \le_m (q|p)$ then $qp \le sr$ and $q \vee p' \le s \vee r'$, $sr \le qp$ and $s \vee r' \le q \vee p'$. So $qp = sr$ and $q \vee p' = s \vee r'$. So $qp = sr$ and $q'p = s'r$. So $q = qp \vee qp' = sr \vee s'r = r$. Therefore $(q|p) = (s|r)$. So $\le_m$ is

antisymmetric. Next for transitivity: If $qp \leq sr$ and $sr \leq tu$, and $q \vee p' \leq s \vee r'$ and $s \vee r' \leq t \vee u'$, then $qp \leq tu$ and $q \vee p' \leq t \vee u'$. So $(q|p) \leq_m (t|u)$. So $\leq_m$ is transitive.

Theorem 21: $\mathcal{L}/\mathcal{L}$ is a lattice under the partial ordering $\leq_m$, where the LUB_m and GLB_m are given by

$$LUB_m\ (q|p,s|r) = (qp \vee sr) \mid (qp \vee sr \vee pr),$$
$$GLB_m\ (q|p,s|r) = (qpsr) \mid (qpsr \vee pq' \vee rs').$$

Proof of Theorem 21: Denoting $(qp \vee sr) \mid (qp \vee sr \vee pr)$ by $(t|u)$, it is clear that $(q|p) \leq_m (t|u)$ and $(s|r) \leq_m (t|u)$ since firstly, $qp \leq qp \vee sr = tu$ and similarly $sr \leq tu$, and furthermore $q \vee p' = qp \vee p' \leq qp \vee sr \vee p' \vee r' = qp \vee sr \vee (pr)' = (qp \vee sr) \vee (qp \vee sr)'(pr)' = (qp \vee sr) \vee [(qp \vee sr) \vee pr]' = t \vee u'$; so $q \vee p' \leq t \vee u'$ and similarly $s \vee r' \leq t \vee u'$. Thus $(t|u)$ is an upper bound of both $(q|p)$ and $(s|r)$. Now if $(x|y)$ is any upper bound of both $(q|p)$ and $(s|r)$ it must be shown that $(x|y)$ is also an upper bound of $(t|u)$. Since $(x|y)$ is an upper bound of $(q|p)$, $qp \leq xy$ and $qp \vee p' \leq xy \vee y'$. Similarly $sr \leq xy$ and $sr \vee r' \leq xy \vee y'$. Therefore $tu = qp \vee sr \leq xy$. Furthermore, $(qp \vee p') \vee (sr \vee r') \leq xy \vee y'$. But the left side of this latter inequality is $(qp \vee sr) \vee p' \vee r' = (qp \vee sr) \vee (pr)' = (qp \vee sr) \vee (qp \vee sr)'(pr)' = (qp \vee sr) \vee (qp \vee sr \vee pr)' = t \vee u'$. So $t \vee u' \leq xy \vee y'$. Therefore $(t|u) \leq_m (x|y)$. Thus $(t|u) = LUB_m\ (q|p,s|r)$.

Next, denoting $(qpsr \mid qpsr \vee pq' \vee rs')$ by $(t|u)$, it is clear that $(t|u)$ is a lower bound of both $(q|p)$ and $(s|r)$ because $tu = qpsr \leq qp$ and similarly $tu \leq sr$, and furthermore because $tu \vee u' = qpsr \vee (qpsr \vee pq' \vee rs')' = qpsr \vee (qpsr)'(pq' \vee rs')' = qpsr \vee (pq' \vee rs')' = qpsr \vee (q \vee p')(s \vee r') \leq q \vee (q \vee p') = q \vee p' = qp \vee p'$; and similarly $tu \vee u' \leq sr \vee r'$. Now if $(x|y)$ is any lower bound of both $(q|p)$ and $(s|r)$, then $xy \leq qp$, $xy \vee y' \leq qp \vee p'$, $xy \leq sr$ and $xy \vee y' \leq sr \vee r'$. So $xy \leq (qp)(sr) = tu$. Furthermore, $xy \vee y' \leq (qp \vee p')(sr \vee r') = (q \vee p')(s \vee r')$ and $tu \vee u' = qpsr \vee (qpsr \vee pq' \vee rs')' = qpsr \vee (qpsr)'(pq' \vee rs')' = qpsr \vee (q \vee p')(s \vee r')$. So $xy \vee y' \leq tu \vee u'$. Therefore $(x|y) \leq_m (t|u)$. Thus $(t|u) = GLB_m\ (q|p,s|r)$. This completes the proof of Theorem 21.

The LUB_m and GLB_m of material implication turn out to be the very operations of disjunction and conjunction derived by I. R. Goodman and H. T. Nguyen [22] by different methods and with somewhat different goals. While these operations have their application, they do not appear to be appropriate for purposes of *general* probability logic. For instance, consider the experiment of rolling a single die once. The compound proposition "if the roll is even then it will be a six or if the roll is odd it will be a five" reduces by the Goodman/Nguyen operations to "if the roll is five or six then it will be five or six", which, of course, is certain and so has conditional probability 1. In contrast, according to the

operations of Theorem 2, this compound conditional reduces to "the roll will be five or six" and has probability 2/6 or 1/3, which corresponds nicely with intuition. See also H. Prade and D. Dubois [20] concerning a comparison of these operations.

As mentioned earlier, still another way to express "p entails q" is to say "q is in the equivalence class (sum ideal, filter class) generated by p". In symbols this is $q \in \langle p\rangle$. Now in a Boolean algebra this relation can be expressed in several equivalent ways including $qp = p$, $q \vee p = q$, $q \vee p' = 1$, and $(q|p) = (1|p)$ corresponding to conjunctive, disjunctive, material and conditional implication respectively. However, as has been shown above, these forms are not equivalent in the conditional closure $\mathcal{L}/\mathcal{L}$ of a Boolean algebra $\mathcal{L}$.

The statement "(q|p) entails (s|r)", when extended to conditionals, then becomes "(s|r) is in the equivalence class of conditionals generated by (q|p)". The trouble here is that before an entailment relation is chosen it is not immediately clear what the meaning is of "the equivalence class of conditionals in $\mathcal{L}/\mathcal{L}$ generated by (q|p)". To clarify this matter there is the following:

Definition 12: The equivalence class $\langle q|p\rangle$ of conditional propositions in $\mathcal{L}/\mathcal{L}$ generated by the conditional proposition (q|p) is

$$\langle q|p\rangle = \{(s|r)\colon (q|p) \leq_C (s|r)\}.$$

Note that this is equivalent to $\langle q|p\rangle = \{(s|r)\colon q \vee p' \leq s \vee r'\}$. The equivalence class $\langle q|p\rangle$ generated by (q|p) is the set of all conditional propositions (s|r) which are true given (q|p). That is,

$$(s|r) \in \langle q|p\rangle \quad \text{if and only if} \quad [(s|r)|(q|p)] = [(1|r)|(q|p)], \tag{40}$$

where the right hand side is just another way of writing $(q|p) \leq_C (s|r)$.

Note that it follows from Definition 12 that the conditionals (q|p) and (0|pq') and the simple proposition $(q \vee p')$, which is equal to $(q \vee p' \mid 1)$, all generate the same equivalence class in $\mathcal{L}/\mathcal{L}$, namely $\langle q \vee p'\rangle$. If this seems strange recall that these conditionals are equivalent only when they are wholly true, i.e. certain, not when they are merely possible, i.e., having a non-zero probability.

In view of all the foregoing it appears that entailment of conditionals by conditionals is fairly well described in $\mathcal{L}/\mathcal{L}$ by conditional implication ($\leq_C$) even though the lack of anti-symmetry means that conditional implication in both directions is not the same as

equivalence of two conditionals. As mentioned previously, $\mathcal{L}/\mathcal{L}$ is quasi-ordered by $\leq_c$ because $\leq_c$ is reflective and transitive.

Since conjunctive implication ($\leq_\wedge$) and material implication ($\leq_m$) individually imply conditional implication, these are stronger forms of entailment than conditional implication. Different circumstances will dictate which kind of implication to use.

To end this section on the recurring issue of non-monotonicity, consider a conditional (q|p) and its contrapositive (p'|q'). As pointed out by the author in [3], pp. 221-2, these two conditionals are equivalent when either is wholly true but otherwise they don't even have the same probability: It is easy to show that if $p \leq q$ then $q' \leq p'$ and conversely. Furthermore a conditional (q|p) conditionally implies its contrapositive (p'|q'):

Theorem 22: $(q|p) \leq_c (p'|q')$

Proof of Theorem 22: $(p'|q') \mid (q|p) = (p' \mid q'(q \vee p')) = (p' \mid q'p') = (1 \mid q'p') = (1 \mid q'(q \vee p')) = (1|q') \mid (q|p)$. So by definition, $(q|p) \leq_c (p'|q')$.

In addition, $P(q|p) = P(p'|q')$ if either is 1. But the conditional probability of a conditional is less than or equal to the conditional probability of its contrapositive if and only if either the corresponding premises or conclusions are so ordered in probability. To make this precise there is the following:

Theorem 23: Suppose $P(p) \neq 0 \neq P(q')$. Then $P(q|p) = 1$ if and only if $P(p'|q') = 1$. Otherwise, $P(q|p) \leq P(p'|q')$ if and only if $P(p) \leq P(q')$ if and only if $P(q) \leq P(p')$.

Proof of Theorem 23: In [3], p.222, it is shown that

$$P(p'|q') = P(q|p) + [1 - P(p)/P(q')]\,[1 - P(q|p)] \tag{41}$$

Clearly, if $P(q|p) = 1$ then so is $P(p'|q') = 1$. By symmetry the converse is also true. Furthermore, $P(q|p) \leq P(p'|q')$ if and only if the product of the brackets is non-negative. Since $[1 - P(q|p)] \geq 0$, this is true if and only if $[1 - P(p)/P(q')] \geq 0$, i.e., if and only if $P(p) \leq P(q')$. So $P(q) \leq P(p')$ using $P(p) = 1 - P(p')$ and $P(q') = 1 - P(q)$.

It is important to realize that in any model in which (q|p) is not false, i.e. not both p true and q false, it is also true that (p'|q') is not false, i.e. not both q' is true and p' is false. However, the fact that these conditionals may be undefined (have truth value U) allows their conditional probabilities to be ordered in either way.

The combining of logic with probability is fraught with the danger of contaminating absolute (certain) information with partially true information, with absolute nonsense

being the result. For instance, it is known with certainty (by definition) that "if an animal is a penguin then it is a bird" and it is also known with very high probability that "if an randomly chosen animal is a bird then it will fly". Therefore one might conclude with high probability that "if an animal is a penguin then it will fly." Such examples should give pause to those who would cavalierly fuse data with suboptimal techniques.

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